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Approximation Numbers in Some Weighted Function Spaces

DOROTHEE HAROSKE

Mathematisches Institut, UHH 17.OG, Fakultät für Mathematik und Informatik, Friedrich-Schiller-Universität, D-07740 Jena, Germany

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In this paper we study weighted function spaces of type $B^s_{p,q}(\mathbb{R}^n, w(x))$ and $F^s_{p,q}(\mathbb{R}^n, w(x))$ where $w(x)$ is a weight function of at most polynomial growth, preferably $w(x) = (1+|x|^2)^{\alpha/2}$ with $\alpha \in \mathbb{R}$. The main result deals with estimates for the approximation numbers of compact embeddings between spaces of this type. Furthermore we are concerned with the dependence of the approximation numbers a_k of compact embeddings between function spaces $B^s_{p,q}(\Omega)$ and $F^s_{p,q}(\Omega)$ on an underlying domain Ω. © 1995 Academic Press, Inc.

1. INTRODUCTION

In [4] and [5] entropy and approximation numbers of compact embeddings between function spaces of type $B^s_{p,q}$ and $F^s_{p,q}$, $s \in \mathbb{R}$, $0 < p \leqslant \infty$ (with $p < \infty$ in the F-case), $0 < q \leqslant \infty$, on a bounded domain Ω in $\mathbb{R}^n$ were thoroughly investigated. Recall that these two scales of spaces cover many well-known classical spaces such as (fractional) Sobolev spaces, Hölder–Zygmund spaces, Besov spaces and (inhomogeneous) Hardy spaces. In [7] we extended these results in some sense, i.e. we studied weighted function spaces of type $B^s_{p,q}(\mathbb{R}^n, w(x))$ and $F^s_{p,q}(\mathbb{R}^n, w(x))$ where $w(x)$ is an admissible weight function of at most polynomial growth, that is a smooth function with

$$0 < w(x) \leqslant c w(y) \langle x - y \rangle^{\alpha} \tag{1}$$

for some $\alpha \geqslant 0$, some $c > 0$ and all $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$. As usual $\langle x \rangle = (1+|x|^2)^{1/2}$. The main result of [7] dealt with relatively sharp estimates for

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the entropy numbers of compact embeddings between function spaces of such type

$$F^{s_1}_{p_1,q_1}(\mathbb{R}^n, w(x)\langle x\rangle^{\beta}) \quad \text{into} \quad F^{s_2}_{p_2,q_2}(\mathbb{R}^n, w(x)) \quad \text{with } \beta>0 \tag{2}$$

and their B-counterparts.

We applied these results in [8] to eigenvalue distributions of pseudo-differential operators. In the present paper we return to the study of the compactness of embeddings of type (2) for its own sake estimating the related approximation numbers.

Weighted spaces of the above and more general type have already been treated before, especially by H.-J. Schmeisser and H. Triebel in [12: 5.1]. Nevertheless we sketched new shorter proofs for some relevant facts in [7] relying not very much on former results.

The plan of the paper is as follows. In Sect. 2 we introduce the spaces $B^s_{p,q}(\mathbb{R}^n, w(x))$ and $F^s_{p,q}(\mathbb{R}^n, w(x))$. We collect some recently proved results which will be of great service for us later on. In particular, we remind the reader of the equivalence of the quasi-norms

$$\|f \mid F^s_{p,q}(\mathbb{R}^n, w(x))\| \quad \text{and} \quad \|wf \mid F^s_{p,q}(\mathbb{R}^n)\| \tag{3}$$

and their B-counterparts. Furthermore recall that for $-\infty<s_2<s_1<\infty$, $0<p_1\leqslant p_2<\infty$, $0<q_1\leqslant\infty$ and $0<q_2\leqslant\infty$, the embedding

$$F^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_1(x)) \quad \text{into} \quad F^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_2(x)) \tag{4}$$

(and its B-counterpart) is compact if and only if

$$s_1-\frac{n}{p_1}>s_2-\frac{n}{p_2} \quad \text{and} \quad \frac{w_2(x)}{w_1(x)}\to 0 \quad \text{as } |x|\to\infty. \tag{5}$$

Finally we mention a helpful weak type embedding

$$B^s_{p,q}(\mathbb{R}^n, \langle x\rangle^{\alpha}) \quad \text{into} \quad weak\text{-}B^s_{p_0,q}(\mathbb{R}^n) \tag{6}$$

where $\alpha>0$ and $1/p_0=1/p+\alpha/n$.

Turning to the entropy and approximation numbers we refer to the respective estimates related to function spaces on domains published in [4] and [5]. In Sect. 3 we regard as a preparation the dependence of the approximation numbers on the certain domain Ω on which function spaces $F^s_{p,q}(\Omega)$ and $B^s_{p,q}(\Omega)$ are defined. Afterwards we state our main theorem. Sect. 4 contains all the proofs.

Unimportant constants are denoted by c, occasionally with additional subscript within the same formula or the same step of the proof. Furthermore, $(k.l/m)$ refers to formula (m) in subsection $k.l$, whereas (j) means formula (j) in the same subsection. In a similar way we quote definitions, propositions and theorems.

2. Definitions and Preliminaries

2.1. *Weighted Function Spaces*

Let $\mathbb{R}^n$ be the Euclidean n-space. We introduce the notation $\langle x \rangle = (1+|x|^2)^{1/2}$ on $\mathbb{R}^n$.

Definition 1. The class of *admissible weight functions* is the collection of all positive C^∞ functions $w(x)$ on $\mathbb{R}^n$ with the following properties:

(i) for any multiindex γ there exists a positive constant c_γ with

$$|D^\gamma w(x)| \leqslant c_\gamma w(x) \qquad \text{for all} \quad x \in \mathbb{R}^n, \tag{1}$$

(ii) there exist two constants $c>0$ and $\alpha \geqslant 0$ such that

$$0 < w(x) \leqslant c w(y) \langle x-y \rangle^\alpha \qquad \text{for all} \quad x \in \mathbb{R}^n \text{ and } y \in \mathbb{R}^n. \tag{2}$$

Remark 1. From (2) it can be easily seen that for suitable constants $c_1>0$ and $c_2>0$ it holds

$$c_1 w(y) \leqslant w(x) \leqslant c_2 w(y) \qquad \text{for all} \quad x \in \mathbb{R}^n,\ y \in \mathbb{R}^n \quad \text{with } |x-y| \leqslant 1. \tag{3}$$

On the other hand we have for admissible weight functions $w_1(x)$ and $w_2(x)$ that both $w_1(x)\, w_2(x)$ and $w_1^{-1}(x)$ are admissible weight functions, too.

Remark 2. We want to explain briefly that the apparently restrictive assumption for $w(x)$ to be a C^∞ function is in fact almost none. Let $w(x)$ be a measurable function in $\mathbb{R}^n$ satisfying (2) and assume $h(x) \geqslant 0$ to be a C^∞-function in $\mathbb{R}^n$, supported by the unit ball with, say, $\int h(x)\, dx = 1$. In other words, $h(x)$ is a so-called mollifier. Then $(h * w)(x)$ defined by

$$(h * w)(x) = \int h(x-y)\, w(y)\, dy \tag{4}$$

is an admissible weight function according to the above definition. As w and $h * w$ are equivalent to each other this finally justifies to concentrate only on smooth representatives without loss of generality.

Now we will briefly remind the reader of the well-known spaces $B^s_{p,q}$ and $F^s_{p,q}$ because we want to define their weighted counterparts afterwards. All spaces in this paper are defined on $\mathbb{R}^n$ and so we omit "$\mathbb{R}^n$" in the sequel. The Schwartz space S and its dual S' of all complex-valued tempered distributions have the usual meaning here. Furthermore, L_p with $0<p\leqslant\infty$, is the usual quasi-Banach space with respect to the Lebesgue measure, quasi-normed by $\|\cdot \mid L_p\|$.

Let $\varphi\in S$ be such that

$$\operatorname{supp}\varphi\subset\{y\in\mathbb{R}^n : |y|<2\} \qquad \text{and} \qquad \varphi(x)=1 \quad \text{if } |x|\leqslant 1, \tag{5}$$

let $\varphi_j(x)=\varphi(2^{-j}x)-\varphi(2^{-j+1}x)$ for $j\in\mathbb{N}$ and put $\varphi_0=\varphi$. Then since $1=\sum_{j=0}^{\infty}\varphi_j(x)$ for all $x\in\mathbb{R}^n$, the $\{\varphi_j\}$ form a dyadic resolution of unity. Given any $f\in S'$, we denote by $\hat{f}$ and $f^\vee$ its Fourier transform and its inverse Fourier transform, respectively. Thus $(\varphi_j\hat{f})^\vee$ is an analytic function on $\mathbb{R}^n$. Based on the unweighted spaces L_p on $\mathbb{R}^n$ we introduce their weighted generalizations $L_p(w(x))$, quasi-normed by

$$\|f \mid L_p(w(\cdot))\| = \|wf \mid L_p\|, \tag{6}$$

where $w(x)>0$ is an (admissible) weight function on $\mathbb{R}^n$ and $0<p\leqslant\infty$.

DEFINITION 2. Let $w(x)$ be an admissible weight function in the sense of Definition 1. Let $s\in\mathbb{R}$, $0<q\leqslant\infty$ and let $\{\varphi_j\}$ be the above dyadic resolution of unity.

(i) Let $0<p\leqslant\infty$. The space $B^s_{p,q}(w(x))$ is the collection of all $f\in S'$ such that

$$\|f \mid B^s_{p,q}(w(\cdot))\| = \left(\sum_{j=0}^{\infty} 2^{jsq}\, \|(\varphi_j\hat{f})^\vee \mid L_p(w(\cdot))\|^q\right)^{1/q} \tag{7}$$

(with the usual modification if $q=\infty$) is finite.

(ii) Let $0<p<\infty$. The space $F^s_{p,q}(w(x))$ is the collection of all $f\in S'$ such that

$$\|f \mid F^s_{p,q}(w(\cdot))\| = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq}\, |(\varphi_j\hat{f})^\vee(\cdot)|^q\right)^{1/q} \,\Big|\, L_p(w(\cdot))\right\| \tag{8}$$

(with the usual modification if $q=\infty$) is finite.

(iii) Let $w(x)=\langle x\rangle^\alpha$ for some $\alpha\in\mathbb{R}$. Then we put

$$B^s_{p,q}(\alpha)=B^s_{p,q}(\langle x\rangle^\alpha) \qquad \text{with} \quad B^s_{p,q}=B^s_{p,q}(0) \tag{9}$$

and

$$F^s_{p,q}(\alpha) = F^s_{p,q}(\langle x \rangle^\alpha) \qquad \text{with} \quad F^s_{p,q} = F^s_{p,q}(0). \tag{10}$$

Remark 3. The theory of the unweighted spaces $B^s_{p,q}$ and $F^s_{p,q}$ has been developed in [13] and [14]. Extending this theory to the above weighted classes of function spaces causes no difficulty. Furthermore, in [12: 5.1] spaces of type $B^s_{p,q}(w(x))$ and $F^s_{p,q}(w(x))$ were investigated in the framework of ultra-distributions for much larger classes of admissible weight functions. Nevertheless also the later developments in the theory of the unweighted spaces $B^s_{p,q}$ and $F^s_{p,q}$, see, e.g., [14], have their more or less obvious counterparts for weighted spaces in the above sense.

Remark 4. Likewise to the unweighted case the above two weighted scales $B^s_{p,q}(w(x))$ and $F^s_{p,q}(w(x))$ cover many other spaces such as weighted (fractional) Sobolev spaces, weighted classical Besov spaces and weighted Hölder–Zygmund spaces. We refer to [12: 5.1] and the literature mentioned there.

2.2. *Embeddings*

In this section we want to collect some important results associated with our topic which have been proved in recent papers, see [7] and the references given there.

PROPOSITION 1. *Let $s \in \mathbb{R}$, $0 < q \leqslant \infty$ and $0 < p \leqslant \infty$ (with $p < \infty$ in the F-case).*

(i) *$B^s_{p,q}(w(x))$ and $F^s_{p,q}(w(x))$ are quasi-Banach spaces (Banach spaces if $p \geqslant 1$ and $q \geqslant 1$), and they are independent of the chosen dyadic resolution of unity $\{\varphi_j\}$.*

(ii) *The operator $f \mapsto wf$ is an isomorphic mapping from $B^s_{p,q}(w(x))$ onto $B^s_{p,q}$ and from $F^s_{p,q}(w(x))$ onto $F^s_{p,q}$. Especially,*

$$\|wf \mid B^s_{p,q}\| \qquad \text{is an equivalent quasi-norm in } B^s_{p,q}(w(x)) \tag{1}$$

and

$$\|wf \mid F^s_{p,q}\| \qquad \text{is an equivalent quasi-norm in } F^s_{p,q}(w(x)). \tag{2}$$

Remark 1. A new short proof of this proposition may be found in [7: 5.1]. Nevertheless there are some other, more complicated proofs and forerunners, e.g., in [12: 5.1] or [6].

Using the above proposition we could immediately extend the embedding theory developed in [13: 2.3.2 and 2.7.1] to the weighted spaces under consideration here if only one weight function is involved. On the other hand we have also regarded in [7] embeddings with different weights. Related to the F-spaces this result reads as follows.

PROPOSITION 2. *Let $w_1(x)$ and $w_2(x)$ be admissible weight functions and*

$$-\infty < s_2 < s_1 < \infty, \quad 0 < p_1 \leqslant p_2 < \infty, \quad 0 < q_1 \leqslant \infty \quad \text{and} \quad 0 < q_2 \leqslant \infty. \tag{3}$$

(i) *Then $F^{s_1}_{p_1, q_1}(w_1(x))$ is continuously embedded in $F^{s_2}_{p_2, q_2}(w_2(x))$,*

$$F^{s_1}_{p_1, q_1}(w_1(x)) \subset F^{s_2}_{p_2, q_2}(w_2(x)), \tag{4}$$

if and only if

$$s_1 - \frac{n}{p_1} \geqslant s_2 - \frac{n}{p_2} \quad \text{and} \quad \frac{w_2(x)}{w_1(x)} \leqslant c < \infty \tag{5}$$

for some $c > 0$ and all $x \in \mathbb{R}^n$.

(ii) *The embedding (4) is compact if and only if*

$$s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2} \quad \text{and} \quad \frac{w_2(x)}{w_1(x)} \to 0 \quad \text{if} \quad |x| \to \infty. \tag{6}$$

Remark 2. A proof of this theorem is given in [7: 5.2]. Obviously one can extend the above proposition to the B-scale. Then p_2 may be infinite and the interesting weighted Hölder–Zygmund spaces $\mathscr{C}^s(w(x)) = B^s_{\infty,\infty}(w(x))$ are included.

In the following we will specify our situation in some sense. Let w_1 and w_2 be two admissible weight functions in the sense of Definition 2.1/1. Then w_1/w_2 is an admissible weight function, too, and Proposition 1 tells us

$$\|f \mid F^s_{p,q}(w_1(\cdot))\| \sim \left\| w_2 f \,\middle|\, F^s_{p,q}\left(\frac{w_1}{w_2}(\cdot)\right)\right\| \tag{7}$$

(equivalent quasi-norms), i.e. $f \mapsto w_2 f$ is an isomorphic mapping from $F^s_{p,q}(w_1(x))$ onto $F^s_{p,q}((w_1/w_2)(x))$ where $w_1(x)$ is assumed to be an admissible weight function. The same holds in the B-case. Studying continuous or compact embeddings it is sufficient to investigate it, without loss of generality, for $w_2(x) = 1$. In the sequel we put $w_1(x) = w(x)$ and specify $w(x) = \langle x \rangle^\alpha$ for some $\alpha > 0$.

To finish this subsection we formulate a weak type continuous embedding assertion. Let $L_{p,\infty} = L_{p,\infty}(\mathbb{R}^n)$ with $0 < p < \infty$ be the usual Lorentz space (Marcinkiewicz space) on $\mathbb{R}^n$ with respect to the Lebesgue measure, see [15: 1.18.6] or [1: p. 216] for definitions.

DEFINITION 3. Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leqslant \infty$. Let $\{\varphi_j\}$ be a dyadic resolution of unity. Then *weak*-$B^s_{p,q}$ is the collection of all $f \in S'$ such that

$$\|f \mid weak\text{-}B^s_{p,q}\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \, \|(\varphi_j \hat{f})^{\vee} \mid L_{p,\infty}\|^q \right)^{1/q} \tag{8}$$

(with the usual modification if $q = \infty$) is finite. Similarly, *weak*-$F^s_{p,q}$ is the collection of all $f \in S'$ such that

$$\|f \mid weak\text{-}F^s_{p,q}\| = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} \, |(\varphi_j \hat{f})^{\vee}(\cdot)|^q \right)^{1/q} \Big| \, L_{p,\infty} \right\| \tag{9}$$

(with the usual modification if $q = \infty$) is finite.

Remark 3. It would also be possible to replace $L_{p,\infty}$ by the more general Lorentz spaces $L_{p,u}$, $0 < p \leqslant \infty$ ($p < \infty$ in the F-case) and $0 < u \leqslant \infty$.

PROPOSITION 3. (i) *Under the restrictions for s, p and q in the above definition both weak-$B^s_{p,q}$ and weak-$F^s_{p,q}$ are quasi-Banach spaces (Banach spaces if $p \geqslant 1$ and $q \geqslant 1$) and they are independent of the chosen dyadic resolution of unity $\{\varphi_j\}$.*

(ii) *Let $s \in \mathbb{R}$, $0 < q \leqslant \infty$, $0 < p \leqslant \infty$ ($p < \infty$ in the F-case), $\alpha > 0$ and $1/p_0 = 1/p + \alpha/n$. Then*

$$B^s_{p,q}(\alpha) \subset weak\text{-}B^s_{p_0,q} \quad \text{and} \quad F^s_{p,q}(\alpha) \subset weak\text{-}F^s_{p_0,q}. \tag{10}$$

Remark 4. A very short proof of the above proposition is included in [7: 2.4].

2.3. *Entropy and Approximation Numbers*

Let B_1 and B_2 be two complex quasi-Banach spaces and let T be a linear and continuous operator from B_1 into B_2. If T is compact then for any given $\varepsilon > 0$ there are finitely many balls in B_2 of radius ε which cover the image TU_1 of the unit ball $U_1 = \{a \in B_1 : \|a \mid B_1\| \leqslant 1\}$.

DEFINITION 1. Let $k \in \mathbb{N}$ and assume $T: B_1 \to B_2$ to be the above continuous operator.

(i) The kth *entropy number* e_k of T is the infimum of all numbers $\varepsilon > 0$ such that there exist 2^{k-1} balls in B_2 of radius ε which cover TU_1.

(ii) The kth *approximation number* a_k of T is the infimum of all numbers $\|T - A\|$ where A runs through the collection of all continuous linear maps from B_1 to B_2 with rank $A < k$.

Remark 1. For details and properties of entropy and approximation numbers we refer to [2], [3], [9] and [11] (always restricted to the case of Banach spaces). There is no difficulty to extend these properties to quasi-Banach spaces.

Similarly to the previous subsection we will collect some recent, already known results which will later on turn out to be the basis for the main result of this paper. We will remind the reader of the papers [4] and [5] concerning entropy and approximation numbers in (unweighted) function spaces on domains.

Before quoting that result we briefly recall the definition of function spaces on domains which are the subject of the succeeding proposition.

DEFINITION 2. Let Ω be a bounded domain in $\mathbb{R}^n$ with C^∞ boundary $\partial\Omega$. Assume $-\infty < s < \infty$, $0 < p \leqslant \infty$ ($p < \infty$ in the F-case) and $0 < q \leqslant \infty$. Then $B^s_{p,q}(\Omega)$ and $F^s_{p,q}(\Omega)$ are the restrictions of $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$, respectively, to Ω.

We denote by $a_+ = \max(0, a)$ for $a \in \mathbb{R}$. Furthermore we always use $a_k \sim k^{-\varrho}$ in the sense that there exist two positive numbers c_1 and c_2 such that

$$c_1 k^{-\varrho} \leqslant a_k \leqslant c_2 k^{-\varrho} \qquad \text{for all} \quad k \in \mathbb{N}. \tag{1}$$

PROPOSITION. *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^∞ boundary $\partial\Omega$. Assume*

$$-\infty < s_2 < s_1 < \infty, \qquad p_1, p_2, q_1, q_2 \in (0, \infty] \tag{2}$$

and suppose that

$$\delta^+ := s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0. \tag{3}$$

Let e_k be the kth entropy number of the natural embedding id: $B^{s_1}_{p_1,q_1}(\Omega) \to B^{s_2}_{p_2,q_2}(\Omega)$ and a_k its kth approximation number.

(i) *Then it holds*

$$e_k \sim k^{-(s_1-s_2)/n}. \tag{4}$$

(ii) *Suppose that in addition to the general hypotheses*

$$\text{either } 0<p_1 \leqslant p_2 \leqslant 2 \text{ or } 2 \leqslant p_1 \leqslant p_2 \leqslant \infty \text{ or } 0<p_2 \leqslant p_1 \leqslant \infty \tag{5}$$

is satisfied. Then it holds

$$a_k \sim k^{-\delta^+/n}. \tag{6}$$

(iii) *Suppose that in addition to the general hypotheses*

$$0<p_1 \leqslant 2 \leqslant p_2 < \infty \quad \text{and} \quad \lambda = \frac{s_1-s_2}{n} - \max\left(\frac{1}{2}-\frac{1}{p_2}, \frac{1}{p_1}-\frac{1}{2}\right) > \frac{1}{2}. \tag{7}$$

Then it holds

$$a_k \sim k^{-\lambda}. \tag{8}$$

(iv) *Suppose that in addition to the general hypotheses*

$$0<p_1 \leqslant 2 \leqslant p_2 \leqslant \infty. \tag{8}$$

Then there are positive constants c_1 and c_2 such that for all $k \in \mathbb{N}$

$$c_1 k^{-\lambda} \leqslant a_k \leqslant c_2 k^{-\delta^+/n} \tag{10}$$

where λ has the same meaning as in (7).

Remark 2. The proposition and its proof will be found in [4] and [5]. Obviously, via the elementary embedding

$$B^s_{p,u} \subset F^s_{p,q} \subset B^s_{p,v} \quad \text{if and only if } u \leqslant \min(p,q) \text{ and } v \geqslant \max(p,q) \tag{11}$$

the above proposition holds also in the F-case, now with $p_1 < \infty$ and $p_2 < \infty$. (There is a new short proof for the "only if"-part of (11) in [7: 4.3].)

Remark 3. The thin lines in the above diagrams Figs. 1–3 shall indicate the different level lines on which the exponents of $k \in \mathbb{N}$ are constant. Fig. 1 refers to e_k whereas Figs. 2 and 3 are related to a_k. In Fig. 2 we made use of the convention $p_1' = \infty$ if $p_1 \leqslant 1$. Then we have for $1/p_1' \leqslant 1/p_2 \leqslant 1/2$ there that $\lambda = 1/2$ is equivalent to $s_2 = s_1 - n/p_1$.

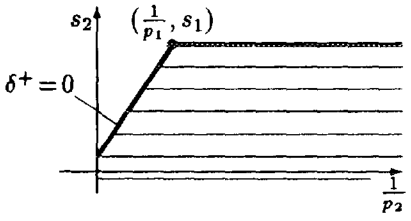

FIGURE 1

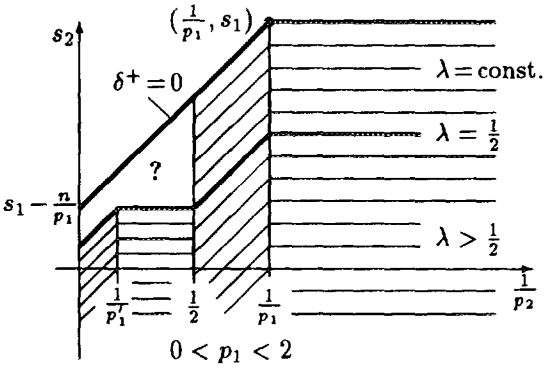

FIGURE 2

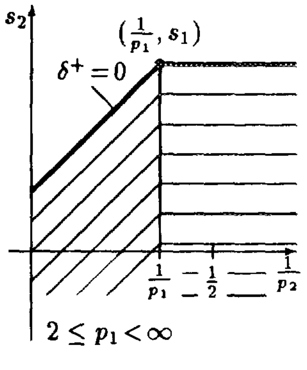

FIGURE 3

3. Approximation Numbers in Weighted Function Spaces

3.1. *Dependence of the Approximation Numbers on Domains*

In this subsection we provide ourselves with a last preparation which may also be regarded as belonging to the proof of the main theorem. But just this proof will already become long enough therefore we prove the following lemma separately and in advance.

LEMMA. *Let* $K_R = \{x \in \mathbb{R}^n : |x| < R\}$, $R \geqslant 1$, *be a ball in* $\mathbb{R}^n$ *centered at the origin. Assume*

$$-\infty < s_2 < s_1 < \infty, \quad 0 < p_1 \leqslant p_2 < \infty \quad \text{and} \quad s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2}. \tag{1}$$

Let a_k^R *be the* k*th approximation number of the compact embedding* $id\colon F^{s_1}_{p_1,q_1}(K_R) \to F^{s_2}_{p_2,q_2}(K_R)$ *with* $a_k = a_k^1$, $k \in \mathbb{N}$. *Then there exist positive constants* c_1 *and* c_2 *such that for* $k \in \mathbb{N}$ *and* $R > 1$ *we have*

$$a^R_{c_1 R^n k} \leqslant c_2 a_k. \tag{2}$$

Remark 1. The above lemma will be proved in 4.1. We introduced the function spaces on domains in Definition 2.3/2. We always put $a_\lambda = a_{[\lambda]}$ if $\lambda \geqslant 1$ and $[\lambda]$ is the largest integer with $[\lambda] \leqslant \lambda$.

COROLLARY. *Let* $A_m = \{x \in \mathbb{R}^n : 2^{m-1} < |x| < 2^{m+1}\}$, $m \in \mathbb{N}$, *be the usual annuli and* $a_k^{(j)}$ *the respective* k*th approximation number of the embedding* $id^{(j)}\colon F^{s_1}_{p_1,q_1}(A_j) \to F^{s_2}_{p_2,q_2}(A_j)$ *where again* (1) *is assumed to be satisfied. Then there exist positive constants* c_1 *and* c_2 *such that for all* $k \in \mathbb{N}$ *and* $j \in \mathbb{N}$ *we get*

$$a^{(j)}_{c_1 2^{jn} k} \leqslant c_2 a_k. \tag{3}$$

Remark 2. The proof is essentially the same as for the above lemma and will not be repeated here. We have to replace $R > 1$ by 2^j, $j \in \mathbb{N}$, then.

3.2. *The Main Theorem*

As we already announced in the beginning the main subject of this paper is to study the approximation numbers of the compact embeddings

$$id^B\colon B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2} \tag{1}$$

and

$$id^F\colon F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2} \tag{2}$$

where the spaces have been introduced in Definition 2.1/2. We also mentioned that this covers the apparently more general cases where the unweighted spaces on the right-hand side of (1) and (2) are replaced by $B^{s_2}_{p_2,q_2}(\beta)$ and $F^{s_2}_{p_2,q_2}(\beta)$, respectively, for some $\beta < \alpha$. One can furthermore imagine to mix B- and F-spaces in (1) and (2) but we give up this possibility. Moreover, it turns out that the third indices never play any role such that we can formulate the theorem for the B-case only and afterwards, via the weighted counterpart of (2.3/11), also the F-case is covered.

Let for $1 \leqslant p \leqslant \infty$ the numbers p' be defined by $1/p + 1/p' = 1$, for $0 < p < 1$ we put $p' = \infty$. Assume that

$$\begin{gathered} -\infty < s_2 < s_1 < \infty, \quad \alpha > 0, \qquad 0 < p_1 < \infty, \quad 0 < q_1 \leqslant \infty, \\ \frac{1}{p_0} = \frac{1}{p_1} + \frac{\alpha}{n}, \qquad p_0 < p_2 < \infty, \quad 0 < q_2 \leqslant \infty \end{gathered} \tag{3}$$

and

$$\delta = s_1 - \frac{n}{p_1} - \left(s_2 - \frac{n}{p_2} \right) > 0. \tag{4}$$

In the usual $(1/p, s)$-diagram we introduce the following regions (see Figs. 4–6):

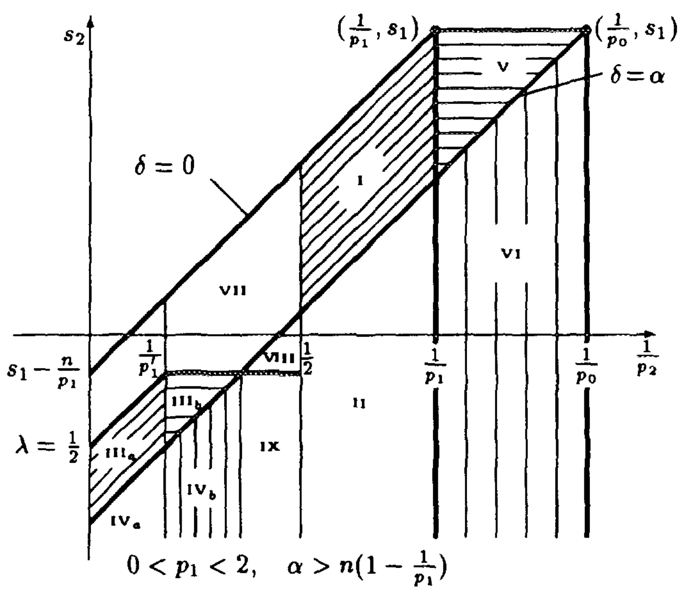

FIGURE 4

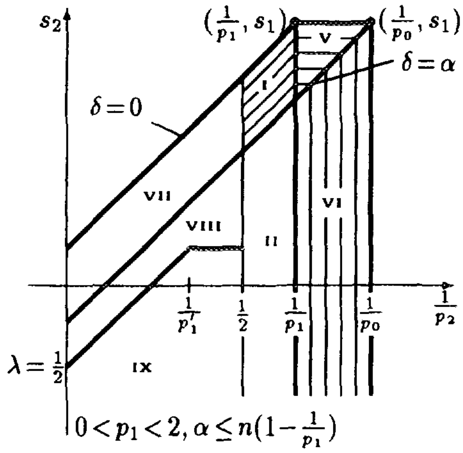

FIGURE 5

I $0<p_1\leqslant p_2\leqslant 2$ or $2\leqslant p_1\leqslant p_2<\infty$, $0<\delta<\alpha$

II $0<p_1\leqslant p_2\leqslant 2$ or $2\leqslant p_1\leqslant p_2<\infty$, $\delta>\alpha$

III $0<p_1<2<p_2<\infty$, $0<\delta<\alpha$,

$$\lambda := \frac{s_1-s_2}{n} - \max\left(\frac{1}{2}-\frac{1}{p_2}, \frac{1}{p_1}-\frac{1}{2}\right) > \frac{1}{2}$$

III$_a$ $0<p_1<2<p_1'\leqslant p_2$, $0<\delta<\alpha$, $\lambda = \frac{\delta}{n}+\frac{1}{p_1}-\frac{1}{2}>\frac{1}{2}$

III$_b$ $0<p_1<2<p_2\leqslant p_1'$, $0<\delta<\alpha$, $\lambda = \frac{\delta}{n}+\frac{1}{2}-\frac{1}{p_2}>\frac{1}{2}$

IV $0<p_1<2<p_2<\infty$, $\delta>\alpha>n\max\left(1-\frac{1}{p_1}, \frac{1}{p_2}\right)$, $\lambda>\frac{1}{2}$

IV$_a$ $0<p_1<2<p_1'\leqslant p_2$, $\delta>\alpha>n\left(1-\frac{1}{p_1}\right)$, $\lambda=\frac{\delta}{n}+\frac{1}{p_1}-\frac{1}{2}>\frac{1}{2}$

IV$_b$ $0<p_1<2<p_2\leqslant p_1'$, $\delta>\alpha>\frac{n}{p_2}$, $\lambda=\frac{\delta}{n}+\frac{1}{2}-\frac{1}{p_2}>\frac{1}{2}$

V $p_0<p_2\leqslant p_1$, $0<\delta<\alpha$

VI $p_0<p_2\leqslant p_1$, $\delta>\alpha$

VII $0<p_1<2<p_2<\infty$, $0<\delta<\alpha$, $\lambda\leqslant\frac{1}{2}$

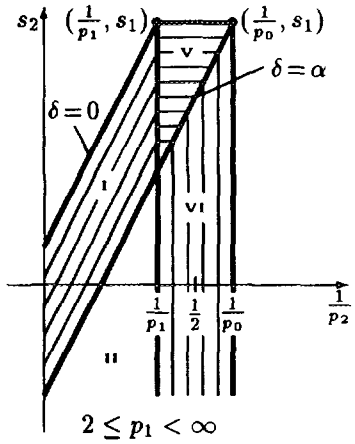

FIGURE 6

VIII $\quad 0<p_1<2<p_2<\infty,\ \alpha<\delta\leqslant n\max\left(1-\dfrac{1}{p_1},\dfrac{1}{p_2}\right)$

IX $\quad 0<p_1<2<p_2<\infty,\ \alpha\leqslant n\max\left(1-\dfrac{1}{p_1},\dfrac{1}{p_2}\right)<\delta.$

THEOREM. *Let a_k be the kth approximation number of the embedding* (1) *and let the assumptions* (3) *and* (4) *be satisfied. Then using the above notations we have the following results*:

(i) *in region* I $\qquad a_k \sim k^{-\delta/n};$ (5)

(ii) *in region* II $\qquad a_k \sim k^{-\alpha/n};$ (6)

(iii) *in region* III, *i.e.* III_a *and* III_b, $\qquad a_k \sim k^{-\lambda};$ (7)

(iv) *in region* IV, *i.e.* IV_a *and* IV_b, *there exist a positive constant c and for any $\varepsilon>0$ a positive constant c_ε such that*

$$ck^{-\alpha/n-\min(1/p_1-1/2,\,1/2-1/p_2)}$$
$$\leqslant a_k \leqslant c_\varepsilon k^{-\alpha/n-\min(1/p_1-1/2,\,1/2-1/p_2)+\varepsilon}; \tag{8}$$

(v) *in region* V $\qquad a_k \sim k^{-(s_1-s_2)/n};$ (9)

(vi) *in region* VI $\qquad a_k \sim k^{-\alpha/n+1/p_2-1/p_1};$ (10)

(vii) *in region* VII *there exist two positive constants c_1 and c_2 such that*

$$c_1 k^{-\delta/n-\min(1/p_1-1/2,\,1/2-1/p_2)} \leqslant a_k \leqslant c_2 k^{-\delta/n}; \tag{11}$$

(viii) *in region* VIII *there exist two positive constants* c_1 *and* c_2 *such that*

$$c_1 k^{-\alpha/n - \min(1/p_1 - 1/2,\, 1/2 - 1/p_2)} \leqslant a_k \leqslant c_2 k^{-\alpha/n}; \tag{12}$$

(ix) *in region* IX *there exist a positive constant* c *and for any* $\varepsilon > 0$ *a positive constant* c_ε *such that*

$$c k^{-\alpha/n - \min(1/p_1 - 1/2,\, 1/2 - 1/p_2)} \leqslant a_k \leqslant c_\varepsilon k^{-\alpha/(2n \max(1 - 1/p_1,\, 1/p_2)) + \varepsilon}. \tag{13}$$

Remark 1. As we emphasized in front of the theorem the results also hold in the F-case.

Remark 2. Depending on the different values for the parameters p_1 and α we indicated in the diagrams Figs. 4–6 the level lines for the corresponding exponents. Concerning the above defined regions VII–IX we omitted this, for looking at (vii)–(ix) in the above theorem the gaps between upper and lower bound appeared too large for having a reasonable intention what the right behaviour of the exponent could be.

Remark 3. Comparing the above theorem with its counterpart (related to entropy numbers) as it is presented in [7: 4.2] we omitted the line "L" where $\delta = \alpha$ in our investigations. Up to now we have not succeeded in developing a separate theory there. Nevertheless we could receive upper or lower bounds for a_k via elementary continuous embeddings and the known behaviour for $\delta > \alpha$ and $\delta < \alpha$. On the other hand we can hardly expect to get a nearly sharp result following that way as Remark 4 below will tell us.

Remark 4. We want to hint at a result of Mynbaev and Otel'baev [10: V, §3, Theorem 9] which in terms of our situation for id: $F^{s_1}_{p_1,2}(\alpha) \to F^0_{p_2,2}$ and with

$$\begin{aligned} &s_1 > 0,\ s_2 = 0, && 1 < p_1 \leqslant p_2 \leqslant 2 \quad \text{or} \quad 2 \leqslant p_1 \leqslant p_2 < \infty, \\ &\alpha > 0, && \delta = s_1 - \frac{n}{p_1} + \frac{n}{p_2} > 0, \end{aligned} \tag{14}$$

gives that

$$a_k = a_k(id) \sim \begin{cases} k^{-\delta/n}, & 0 < \delta < \alpha \\ \left(\dfrac{k}{\log k}\right)^{-\alpha/n}, & \delta = \alpha,\ k \geqslant k_0 \\ k^{-\alpha/n}, & \delta > \alpha. \end{cases} \tag{15}$$

The compatibility of our results and those in the cases $0 < \delta < \alpha$ and $\delta > \alpha$ is the best possible one, namely coincidence. Therefore we should also look for estimates similar to the above ones in the case $\delta=\alpha$. Although the used methods to prove (15) in [10] are completely different from ours we take (15) for granted and try to find a generalization in our sense, i.e. $-\infty<s_2<s_1<\infty$, $0<p_1\leqslant p_2\leqslant 2$ or $0<p_2\leqslant p_1<\infty$, $0<q_1\leqslant\infty$ and $0<q_2\leqslant\infty$. Remembering the situation for the e_k's in [7: 4.2] a dependence on the third indices may well happen. In (15) we have $q_1=q_2=2$ and thus a possible influence could have disappeared.

4. Proofs

4.1. *Proof of Lemma* 3.1

In the sequel we will denote by $\hat{p}=\min(1,p)$ for any p, $0<p<\infty$.

Proof. *Step* 1. As a preparation we first investigate a special open set $\Omega\subset\mathbb{R}^n$, defined as

$$\Omega=\bigcup_{j=1}^{N} K^{(j)}, \qquad \overline{K^{(j)}}\cap\overline{K^{(l)}}=\varnothing, \quad j\neq l \tag{1}$$

where $N\in\mathbb{N}$ is arbitrary and $\{K^{(j)}\}_{j=1}^{N}$ are shifted open unit balls. As usual, $\bar{A}$ means the closure of an open set A. The idea behind is first to handle this simpler case above, i.e. to estimate the respective approximation numbers $a_k^{(\Omega)}$ by a_k and afterwards to cover K_R by finitely many such Ω's from (1).

Let $u\in F^{s_1}_{p_1,q_1}(\Omega)$, then, in a slight abuse of notations,

$$u=\sum_{j=1}^{N} u_j \qquad \text{with} \quad u_j\in F^{s_1}_{p_1,q_1}(K^{(j)}) \tag{2}$$

and, by definition,

$$\|u \mid F^{s_1}_{p_1,q_1}(\Omega)\|^{\widehat{p_1}}=\sum_{j=1}^{N}\|u_j \mid F^{s_1}_{p_1,q_1}(K^{(j)})\|^{\widehat{p_1}} \tag{3}$$

to adapt it to the localization principle for $F^s_{p,q}$-spaces, see [14: 2.4.7], used in the second step.

Let $\varepsilon>0$ and choose $T_j\in L(F^{s_1}_{p_1,q_1}(K^{(j)})\to F^{s_2}_{p_2,q_2}(K^{(j)}))$ such that

$$\operatorname{rank} T_j\leqslant r, \qquad j=1,\ldots,N, \tag{4}$$

and

$$\|u_j-T_ju_j \mid F^{s_2}_{p_2,q_2}(K^{(j)})\|^{\widehat{p_2}}\leqslant(1+\varepsilon)^{\widehat{p_2}}a_r^{\widehat{p_2}}\,\|u_j \mid F^{s_1}_{p_1,q_1}(K^{(j)})\|^{\widehat{p_2}} \tag{5}$$

where we additionally used $a_k^{(K^{(j)})} = a_k$, $k \in \mathbb{N}$, for those (shifted) open unit balls $K^{(j)}$.

Let $T = \sum_{j=1}^{N} T_j$ be such that

$$Tu = \sum_{j=1}^{N} T_j \left(\sum_{l=1}^{N} u_l \right) =: \sum_{j=1}^{N} T_j u_j. \tag{6}$$

Then it holds

$$\begin{aligned} \|u - Tu \mid F^{s_2}_{p_2, q_2}(\Omega)\|^{\widehat{p_2}} &= \sum_{j=1}^{N} \|u_j - T_j u_j \mid F^{s_2}_{p_2, q_2}(K^{(j)})\|^{\widehat{p_2}} \\ &\leqslant (1+\varepsilon)^{\widehat{p_2}} a_r^{\widehat{p_2}} \sum_{j=1}^{N} \|u_j \mid F^{s_1}_{p_1, q_1}(K^{(j)})\|^{\widehat{p_2}} \\ &\leqslant (1+\varepsilon)^{\widehat{p_2}} a_r^{\widehat{p_2}} \|u \mid F^{s_1}_{p_1, q_1}(\Omega)\|^{\widehat{p_2}} \end{aligned} \tag{7}$$

where we used (3), (5), $p_1 \leqslant p_2$ and the special construction of Ω. By (4) and (7) we have for arbitrary small $\varepsilon > 0$

$$\|id_{|\Omega} - T\| \leqslant (1+\varepsilon)\, a_r, \qquad \text{rank } T \leqslant Nr \tag{8}$$

and consequently

$$a_{Nr}^{(\Omega)} \leqslant a_r. \tag{9}$$

Step 2. We consider now the above ball K_R, $R \geqslant 1$, and look for a suitable covering in the sense of Step 1. Let $(1/n)\,\mathbb{Z}^n$ be the lattice such that

$$\theta \in \frac{1}{n}\,\mathbb{Z}^n \Leftrightarrow \exists k \in \mathbb{Z}^n\colon \theta = \frac{1}{n} k \tag{10}$$

holds for every lattice point θ, which means in terms of its coordinates

$$(\theta_1, ..., \theta_n) \in \frac{1}{n}\,\mathbb{Z}^n \Leftrightarrow \exists (k_1, ..., k_n) \in \mathbb{Z}^n\colon \theta_j = \frac{1}{n} k_j, \qquad j = 1, ..., n. \tag{11}$$

Furthermore we have the following sub-lattices $\mathbb{Z}^n_\theta$

$$\mathbb{Z}^n_\theta = \theta + 3\mathbb{Z}^n, \quad \theta \in \frac{1}{n}\,\mathbb{Z}^n, \quad \theta_j \in \left\{0, ..., \frac{3n-1}{n}\right\}, \quad j = 1, ..., n. \tag{12}$$

In other words, any sub-lattice $\mathbb{Z}^n_\theta$ is a shifted $3\mathbb{Z}^n$-lattice which is uniquely specified by its "basis point" θ in the cube $[0, (3n-1)/n]^n$. Thus

$$\sharp \left\{ \theta \in \frac{1}{n}\,\mathbb{Z}^n : 0 \leqslant \theta_j \leqslant \frac{3n-1}{n} \right\} = (3n)^n =: L \tag{13}$$

and obviously

$$\bigcup_{r=1}^{L} \mathbb{Z}^n_{\theta_r} = \frac{1}{n}\mathbb{Z}^n, \qquad \theta_r \in Q, \tag{14}$$

where we introduced the notation

$$Q = \left\{\theta \in \frac{1}{n}\mathbb{Z}^n : 0 \leqslant \theta_j \leqslant \frac{3n-1}{n}, j = 1, ..., n\right\}. \tag{15}$$

Let B^n_r be the following system of translated unit balls

$$B^n_r = \{K(x_l) : x_l \in \mathbb{Z}^n_{\theta_r}\} \tag{16}$$

for $\theta_r \in Q$, $r = 1, ..., L$, and $K(x_l)$ stands for a ball of radius 1 centered at x_l. Consequently (14) and (16) lead to

$$\bigcup_{r=1}^{L} B^n_r = \mathbb{R}^n. \tag{17}$$

Consider a resolution of unity $\varphi = \{\varphi^r_l\}_{l \in \mathbb{Z}^n, r=1,...,L}$, assigned to the balls $K(x_l)$ from (16) such that $\operatorname{supp} \varphi^r_l \subset K(x_l) \in B^n_r$ and

$$\sum_{r=1}^{L} \sum_{l \in \mathbb{Z}^n} \varphi^r_l(x) = 1, \qquad x \in \mathbb{R}^n. \tag{18}$$

Setting

$$\varrho_r = \sum_{l \in \mathbb{Z}^n} \varphi^r_l, \qquad r = 1, ..., L, \tag{19}$$

(18) becomes

$$\sum_{r=1}^{L} \varrho_r(x) = 1, \qquad x \in \mathbb{R}^n \tag{20}$$

and

$$\operatorname{supp} \varrho_r \subset B^n_r. \tag{21}$$

Let $\psi_r \in C^\infty(B^n_r)$ be such that $\operatorname{supp} \psi_r \subset B^n_r$ and

$$\psi_r(x) = 1, \qquad x \in \operatorname{supp} \varrho_r. \tag{22}$$

Let $\varepsilon > 0$ and assume $T_r\colon F^{s_1}_{p_1,q_1}(K_R \cap B^n_r) \to F^{s_2}_{p_2,q_2}(K_R \cap B^n_r)$ an operator on B^n_r, extended by zero outside $B^n_r \cap K_R$, rank $T_r \leqslant k$ and

$$\|(id - T_r)_{|B^n_r \cap K_R}\|^{\widehat{p_2}} \leqslant (1+\varepsilon)^{\widehat{p_2}} (a_k^{(B^n_r \cap K_R)})^{\widehat{p_2}}. \tag{23}$$

Caused by the symmetry of our construction we have for large R

$$\|(id - T_r)_{|B^n_r \cap K_R}\|^{\widehat{p_2}} \leqslant (1+\varepsilon)^{\widehat{p_2}} (a_k^{(B^n_1 \cap K_R)})^{\widehat{p_2}}, \qquad r = 1, ..., L. \tag{24}$$

Let $u \in F^{s_1}_{p_1,q_1}(K_R)$. Thus (20), (22), (24), $p_2 \geqslant p_1$ and the already mentioned localization principle for F-spaces yield

$$\begin{aligned}
&\left\| u - \sum_{r=1}^{L} \psi_r T_r \varrho_r u \,\middle|\, F^{s_2}_{p_2,q_2}(K_R) \right\|^{\widehat{p_2}} \\
&\leqslant c_1 \sum_{r=1}^{L} \|\psi_r \varrho_r u - \psi_r T_r \varrho_r u \mid F^{s_2}_{p_2,q_2}(K_R \cap B^n_r)\|^{\widehat{p_2}} \\
&\leqslant c_2 \sum_{r=1}^{L} \|\varrho_r u - T_r \varrho_r u \mid F^{s_2}_{p_2,q_2}(K_R \cap B^n_r)\|^{\widehat{p_2}} \\
&\leqslant c_3 \sum_{r=1}^{L} \|(id - T_r)_{|B^n_r \cap K_R}\|^{\widehat{p_2}} \cdot \|\varrho_r u \mid F^{s_1}_{p_1,q_1}(K_R \cap B^n_r)\|^{\widehat{p_2}} \\
&\leqslant c_4 (1+\varepsilon)^{\widehat{p_2}} (a_k^{(B^n_1 \cap K_R)})^{\widehat{p_2}} \|u \mid F^{s_1}_{p_1,q_1}(K_R)\|^{\widehat{p_2}}.
\end{aligned} \tag{25}$$

Consequently we have for $T := \sum_{r=1}^{L} T_r$, rank $T \leqslant Lk$,

$$a^R_{Lk} \leqslant c a_k^{(B^n_1 \cap K_R)}. \tag{26}$$

Let N_r be the number of balls $K(x_l)$ belonging to B^n_r which have a non-empty intersection with K_R and put $N := \max\{N_r, r = 1, ..., L\}$. Again for large R we get $N \sim N_r$, $r = 1, ..., L$, and after substituting $k \in \mathbb{N}$ by Nk, (26) becomes

$$a^R_{LNk} \leqslant c_1 a_{Nk}^{(B^n_1 \cap K_R)} \leqslant c_2 a_k \tag{27}$$

where we used Step 1. Furthermore by usual volume arguments we have $LN \sim cR^n$ and so finally

$$a^R_{c_1 R^n k} \leqslant c_2 a_k. \tag{28}$$

4.2. *Proof of the Main Theorem*

We divide the long proof into 7 steps. First we prove the estimates from below. Mainly there exist two different methods: to use respective estimates for approximation numbers in function spaces on domains or to shift the problem to the l_p-situation where one already has such estimates. These first two steps will be the same for the B- and F-spaces. Afterwards we show the sufficiency of proving the upper estimates for the F-spaces as we can then reduce the situation of the B-spaces to that one. We have to follow this rather complicated way as we want to make use of Lemma 3.1 which holds in the F-case only. Caring about the estimates from above the main tool will turn out a tricky partition of $\mathbb{R}^n$ into annuli in connection with the already investigated situation on domains, see Proposition 2.3 and Corollary 3.1.

Proof. *Step* 1. Let $0<\delta<\alpha$ and $0<p_1\leqslant p_2\leqslant 2$ or $2\leqslant p_1\leqslant p_2<\infty$ or $p_0<p_2\leqslant p_1<\infty$, i.e., we handle regions I and V. By the well-known extension-restriction procedure and Proposition 2.2/1 we have for arbitrary smooth bounded domains $\Omega\subset\mathbb{R}^n$

$$a_k(B^{s_1}_{p_1,q_1}(\Omega)\to B^{s_2}_{p_2,q_2}(\Omega))\leqslant c a_k(B^{s_1}_{p_1,q_1}(\alpha)\to B^{s_2}_{p_2,q_2})=c a_k \tag{1}$$

where we additionally used the multiplicativity of approximation numbers. Now recall the already mentioned results for bounded domains, see Proposition 2.3, thus (1) yields

$$a_k\geqslant c k^{-\delta^+/n},\qquad \delta^+=s_1-s_2-n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+. \tag{2}$$

Likewise we handle the situation in the regions III and VII where (2.3/8) and (2.3/10) provide

$$a_k(B^{s_1}_{p_1,q_1}(\Omega)\to B^{s_2}_{p_2,q_2}(\Omega))\geqslant c k^{-\lambda}$$

and consequently

$$a_k\geqslant c k^{-\lambda} \tag{3}$$

with

$$\lambda=\frac{s_1-s_2}{n}-\max\left(\frac{1}{p_1}-\frac{1}{2},\frac{1}{2}-\frac{1}{p_2}\right)=\frac{\delta}{n}+\min\left(\frac{1}{p_1}-\frac{1}{2},\frac{1}{2}-\frac{1}{p_2}\right).$$

Hence we have proved the lower estimates in (i), (iii), (v) and (vii).

Step 2. We are now going to prove the lower estimates of (ii), (iv), (vi), (viii) and (ix). Although this could be similarly done for B- and F-spaces we will concentrate on the F-spaces. Regarding the lower estimates in question one observes that no s-parameters are involved in the exponents. It is only $\delta=s_1-s_2-n(1/p_1-1/p_2)>\alpha$ assumed to hold.

Consequently one can immediately get the estimates in the B-case via the elementary embeddings (2.3/11) and their obvious weighted counterparts

$$B^{s+\varepsilon}_{p,q_0}(\alpha) \subset F^{s}_{p,q_1}(\alpha) \subset B^{s-\varepsilon}_{p,q_2}(\alpha) \tag{4}$$

for $s \in \mathbb{R}$, $\varepsilon > 0$, $0 < p < \infty$, $0 < q_0 \leqslant \infty$, $0 < q_1 \leqslant \infty$, $0 < q_2 \leqslant \infty$, $\alpha > 0$. In detail, the multiplicativity of approximation numbers then yields

$$a_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \geqslant c a_k(F^{s_1+\varepsilon}_{p_1,u_1}(\alpha) \to F^{s_2-\varepsilon}_{p_2,u_2}) \tag{5}$$

where $0 < u_1 \leqslant \infty$, $0 < u_2 \leqslant \infty$ and $\varepsilon > 0$ and thus always $(s_1+\varepsilon) - (s_2-\varepsilon) - n(1/p_1 - 1/p_2) > \alpha$ is satisfied.

We now want to make use of an argumentation similar to that one in [4: 4.3.7] and [5: 4.3.1]. We consider the following commutative diagram

$$\begin{array}{ccc} F^{s_1}_{p_1,q_1}(\alpha) & \xrightarrow{id^F} & F^{s_2}_{p_2,q_2} \\ {\scriptstyle A}\uparrow & & \downarrow{\scriptstyle B} \\ l^{N_j}_{p_1} & \xrightarrow[id_l]{} & l^{N_j}_{p_2} \end{array} \tag{6}$$

where $N_j = 2^{jn}$, id_l is the identity map from $l^{N_j}_{p_1}$ to $l^{N_j}_{p_2}$ and id^F as in (3.2/2). Recall that l^m_p, $m \in \mathbb{N}$, $0 < p < \infty$, is the linear space of all complex m-tuples $y = (y_j)$, furnished with the quasi-norm

$$\|y \mid l^m_p\| = \left(\sum_{j=1}^{m} |y_j|^p \right)^{1/p}.$$

We divide $\mathbb{R}^n$ into the usual annuli $A_j = \{x \in \mathbb{R}^n : 2^{j-1} < |x| < 2^{j+1}\}$ for $j \in \mathbb{N}$. Let $\Phi \in C^\infty(\mathbb{R}^n)$ with $\operatorname{supp} \Phi \subset B_1$, the unit ball, and, say, $\int \Phi(x)\, dx = 1$. Let A be the following operator

$$A: l^{N_j}_{p_1} \to F^{s_1}_{p_1,q_1}(\alpha), \qquad \{\alpha_r\}^{N_j}_{r=1} \mapsto \sum_{r=1}^{N_j} \alpha_r \Phi(x_r - x) \tag{7}$$

where the x_r are those $k \in \mathbb{Z}^n$ such that $x_r = k \in A_j$. Neglecting constants we thus can assume that there are N_j such points. Applying the localization principle for F-spaces, see [14: 2.4.7], we may assume

$$\|A\{\alpha_r\} \mid F^{s_1}_{p_1,q_1}(\alpha)\|^{p_1} \sim 2^{j\alpha p_1} \sum_{r=1}^{N_j} |\alpha_r|^{p_1} \tag{8}$$

for $\langle x \rangle^\alpha \sim 2^{j\alpha}$ in A_j. In other words,

$$\|A\| \leqslant 2^{j\alpha}. \tag{9}$$

Consider now a map $\Psi \in C^\infty(\mathbb{R}^n)$, supp Ψ concentrated near the origin and $\Psi(x)=1$ for $x \in \operatorname{supp} \Phi$. Denote $\Psi_r(x) := \Psi(x_r - x)$, $r = 1, ..., N_j$. Then we put

$$B: F^{s_2}_{p_2, q_2} \to l^{N_j}_{p_2}, \qquad f \mapsto \{(f, \Psi_r)\}^{N_j}_{r=1}. \tag{10}$$

Estimating the norm of B we get

$$\begin{aligned} |(f, \Psi_r)| &= \left| \int f(x)\, \Psi(x_r - x)\, dx \right| \\ &= \left| \int f(x)\, \Psi(x_r - x)\, \Lambda(x_r - x)\, dx \right| \end{aligned} \tag{11}$$

where $\Lambda \in C^\infty(\mathbb{R}^n)$, supp Λ concentrated near the origin and $\Lambda(x)=1$ for $x \in \operatorname{supp} \Psi$. Using $\Lambda_r(x) = \Lambda(x_r - x)$ then (11) becomes

$$\begin{aligned} |(f, \Psi_r)| &= \left| \int (f\Lambda_r)(x)\, \Psi(x_r - x)\, dx \right| = |((f\Lambda_r) * \Psi)(x_r)| \\ &\leqslant \sup_{y \in \mathbb{R}^n} |((f\Lambda_r) * \Psi)(y)| \leqslant \| f\Lambda_r \mid B^\sigma_{\infty, \infty}\| \end{aligned} \tag{12}$$

for any $\sigma \in \mathbb{R}$. This follows from the characterization of these spaces via local means, see [14: 2.5.3]. The elementary embedding $F^{s_2}_{p_2, q_2} \subset B^\sigma_{\infty, \infty}$ for $s_2 - n/p_2 > \sigma$ yields

$$|(f, \Psi_r)| \leqslant c\, \| f\Lambda_r \mid F^{s_2}_{p_2, q_2}\|. \tag{13}$$

Applying again the above mentioned localization principle for F-spaces to (13) we get

$$\sum_{r=1}^{N_j} |(f, \Psi_r)|^{p_2} \leqslant c\, \| f \mid F^{s_2}_{p_2, q_2}\|^{p_2} \tag{14}$$

which provides

$$\|B\| \leqslant c. \tag{15}$$

By construction we have

$$id_l = B \circ id^F \circ A. \tag{16}$$

Hence (9), (15) and the multiplicativity of approximation numbers lead to

$$a_k(id^F) \geqslant c 2^{-j\alpha} a_k(id_l). \tag{17}$$

Concerning $a_k(id_l)$ we make use of [5: 3.2.2 and 3.2.4] which tells us

$$a_k(id_l) \geqslant c2^{jn(1/p_2 - 1/p_1)} \qquad \text{in region VI,} \tag{18}$$

$$a_k(id_l) \sim 1 \qquad \text{in region II,} \tag{19}$$

$$a_k(id_l) \geqslant c2^{-jn \min(1/p_1 - 1/2,\, 1/2 - 1/p_2)} \qquad \text{in IV, VIII and IX} \tag{20}$$

for $k = 2^{jn-1}$. Then (17)–(20) finally result in the estimates from below in (ii), (iv), (vi), (viii) and (ix).

Step 3. We now turn to the estimates from above. First we will show that it is sufficient to deal with the F-case only. Remembering our remark at the beginning of Step 2 this is obvious concerning the regions II, IV, VI, VIII and IX, whereas the upper estimate in VII is a direct consequence of I:

$$a_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \leqslant c a_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_3}_{p_3,q_2})\, a_k(B^{s_3}_{p_3,q_2} \to B^{s_2}_{p_2,q_2}) \tag{21}$$

where we choose p_3 such that $0 < p_1 \leqslant p_3 \leqslant 2$ and $s_3 \in \mathbb{R}$ such that

$$s_3 - \frac{n}{p_3} = s_2 - \frac{n}{p_2}, \qquad s_2 < s_3 < s_1. \tag{22}$$

It remains deriving the cases (i), (iii) and (v) in the B-case from those in the F-case.

We remember again a construction from [4: p. 146/147] where $f \in B^s_{p,q}(\mathbb{R}^n)$ was divided into $f = \sum_{j=0}^{N} (\varphi_j \hat{f})^\vee + \sum_{j=N+1}^{\infty} (\varphi_j \hat{f})^\vee = f_N + f^N$ with $N \in \mathbb{N}$ and $\{\varphi_j\}_{j=0}^{\infty}$ a smooth dyadic partition of unity. Subsequently the above function f_N was splitted up into $f_N = f_{N,1} + f_{N,2}$. We do not want to repeat all the details. We are interested only in the final result that came out : via the above way a linear operator $f \mapsto f - f_{N,1}$ could be constructed approximating the embedding in question in region I. The most important point for us is its linearity which allows us to use interpolation arguments even in that case of approximation numbers. Assume the estimates from above in region I to be true in the F-case, i.e. we have

$$a_k(F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2}) \leqslant ck^{-\delta/n} \tag{23}$$

where $0 < s_1 - s_2 - n(1/p_1 - 1/p_2) < \alpha$, $0 < q_1 \leqslant \infty$, $0 < q_2 \leqslant \infty$. We choose now $\sigma_1 < s_1 < \sigma_2$ such that it holds

$$0 < \delta_1 = \sigma_1 - \frac{n}{p_1} - \left(s_2 - \frac{n}{p_2}\right) < \alpha, \qquad 0 < \delta_2 = \sigma_2 - \frac{n}{p_1} - \left(s_2 - \frac{n}{p_2}\right) < \alpha \tag{24}$$

and

$$s_1 = (1-\theta)\,\sigma_1 + \theta\sigma_2 \tag{25}$$

for some θ, $0<\theta<1$. Then (23) applies also to the embeddings $F^{\sigma_1}_{p_1,u_1}(\alpha)\to F^{s_2}_{p_2,q_2}$ and $F^{\sigma_2}_{p_1,u_2}(\alpha)\to F^{s_2}_{p_2,q_2}$ for arbitrary $0<u_1\leqslant\infty$, $0<u_2\leqslant\infty$. Holding now the target space $F^{s_2}_{p_2,q_2}$ fixed we have for any linear operator T, which maps

$$T\colon F^{\sigma_1}_{p_1,u_1}(\alpha)\to F^{s_2}_{p_2,q_2}, \qquad T\colon F^{\sigma_2}_{p_1,u_2}(\alpha)\to F^{s_2}_{p_2,q_2}$$

that via real interpolation we get

$$T\colon (F^{\sigma_1}_{p_1,u_1}(\alpha), F^{\sigma_2}_{p_1,u_2}(\alpha))_{\theta,q_1}\to (F^{s_2}_{p_2,q_2}, F^{s_2}_{p_2,q_2})_{\theta,q_1}, \tag{26}$$

i.e.

$$T\colon B^{s_1}_{p_1,q_1}(\alpha)\to F^{s_2}_{p_2,q_2} \tag{27}$$

and

$$\begin{aligned}&\|T \mid B^{s_1}_{p_1,q_1}(\alpha)\to F^{s_2}_{p_2,q_2}\|\\ &\quad\leqslant c\,\|T \mid F^{\sigma_1}_{p_1,u_1}(\alpha)\to F^{s_2}_{p_2,q_2}\|^{1-\theta}\,\|T \mid F^{\sigma_2}_{p_1,u_2}(\alpha)\to F^{s_2}_{p_2,q_2}\|^{\theta}.\end{aligned} \tag{28}$$

Here it was essential to have the same target space which then, in fact, is not interpolated. For details concerning the real interpolation of B- and F-spaces see [13: 2.4.2] for the unweighted case. The needed extension to weighted spaces then follows from Proposition 2.2/1(ii). Specializing now T by $f\mapsto f-f_{N,1}$ we have from (23), (24), (25) and (28)

$$a_k(B^{s_1}_{p_1,q_1}(\alpha)\to F^{s_2}_{p_2,q_2})\leqslant ck^{-\delta/n}. \tag{29}$$

Afterwards we repeat the same, now fixing the original space $B^{s_1}_{p_1,q_1}(\alpha)$. In other words, (26) and (27) are then replaced by

$$T\colon (B^{s_1}_{p_1,q_1}(\alpha), B^{s_1}_{p_1,q_1}(\alpha))_{\theta,q_2}\to (F^{\sigma_1}_{p_2,u_1}, F^{\sigma_2}_{p_2,u_2})_{\theta,q_2} \tag{30}$$

and

$$T\colon B^{s_1}_{p_1,q_1}(\alpha)\to B^{s_2}_{p_2,q_2} \tag{31}$$

where we choose $\sigma_1<s_2<\sigma_2$ such that

$$0<\delta_1=s_1-\frac{n}{p_1}-\left(\sigma_1-\frac{n}{p_2}\right)<\alpha,\quad 0<\delta_2=s_1-\frac{n}{p_1}-\left(\sigma_2-\frac{n}{p_2}\right)<\alpha \tag{32}$$

and

$$s_2=(1-\theta)\,\sigma_1+\theta\sigma_2 \tag{33}$$

are satisfied. Consequently we finally get from $\delta=(1-\theta)\,\delta_1+\theta\delta_2$ that

$$a_k(B^{s_1}_{p_1,q_1}(\alpha)\to B^{s_2}_{p_2,q_2})\leqslant ck^{-\delta/n} \tag{34}$$

in region I where always the respective F-result is assumed to hold. In particular we have

$$a_k(B^{s_1}_{p_1,q_1}(\alpha)\to B^{s_2}_{p_1,q_2})\leqslant ck^{-(s_1-s_2)/n},\qquad 0<s_1-s_2<\alpha \tag{35}$$

and $0<q_1\leqslant\infty$, $0<q_2\leqslant\infty$ and that is just the key to cope with the regions III and V. The construction is simple but effective. We always have now $0<\delta=s_1-n/p_1-s_2+n/p_2<\alpha$ and thus can choose $\sigma_1\in\mathbb{R}$ and $\sigma_2\in\mathbb{R}$ such that for some $\alpha_1>0$, $\alpha_2>0$, $\alpha_1+\alpha_2<\alpha$ it holds

$$0<s_1-\sigma_1<\alpha_1,\quad 0<\sigma_1-\frac{n}{p_1}-\sigma_2+\frac{n}{p_2}<\alpha-\alpha_1-\alpha_2,\quad 0<\sigma_2-s_2<\alpha_2. \tag{36}$$

Next we split our embedding $id\colon B^{s_1}_{p_1,q_1}(\alpha)\to B^{s_2}_{p_2,q_2}$ into five:

$$id_1\colon B^{s_1}_{p_1,q_1}(\alpha)\to B^{\sigma_1}_{p_1,\tau_1}(\alpha-\alpha_1) \tag{37}$$

$$id_2\colon B^{\sigma_1}_{p_1,\tau_1}(\alpha-\alpha_1)\to F^{\sigma_1}_{p_1,u_1}(\alpha-\alpha_1) \tag{38}$$

$$id_3\colon F^{\sigma_1}_{p_1,u_1}(\alpha-\alpha_1)\to F^{\sigma_2}_{p_2,u_2}(\alpha_2) \tag{39}$$

$$id_4\colon F^{\sigma_2}_{p_2,u_2}(\alpha_2)\to B^{\sigma_2}_{p_2,\tau_2}(\alpha_2) \tag{40}$$

$$id_5\colon B^{\sigma_2}_{p_2,\tau_2}(\alpha_2)\to B^{s_2}_{p_2,q_2} \tag{41}$$

where $0<\tau_1\leqslant p_1\leqslant u_1<\infty$, $0<u_2\leqslant p_2\leqslant\tau_2<\infty$ and σ_1 and σ_2 as in (36). We apply (35) to id_1 and id_5, note the continuity of (38) and (40) and hence the multiplicativity of approximation numbers provides

$$a_k\leqslant ck^{-(s_1-\sigma_1)/n-(\sigma_2-s_2)/n}a_k(id_3). \tag{42}$$

Assuming now the respective estimates in the F-case to be true, (42) becomes in region III

$$a_k\leqslant ck^{-(s_1-\sigma_1)/n-(\sigma_2-s_2)/n-(\sigma_1-\sigma_2)/n-\max(1/2-1/p_2,\,1/p_1-1/2)}=ck^{-\lambda} \tag{43}$$

and in region V

$$a_k\leqslant ck^{-(s_1-\sigma_1)/n-(\sigma_2-s_2)/n-(\sigma_1-\sigma_2)/n}=ck^{-(s_1-s_2)/n}. \tag{44}$$

Regarding (43) we have only to ensure in region III that $(\sigma_1 - \sigma_2)/n > 1/2 + \max(1/2 - 1/p_2, 1/p_1 - 1/2)$ can always be suitably chosen. In other words, by (36) it is necessary to have

$$\frac{\alpha - \alpha_1 - \alpha_2}{n} + \frac{1}{p_1} - \frac{1}{p_2} > \frac{1}{2} + \max\left(\frac{1}{2} - \frac{1}{p_2}, \frac{1}{p_1} - \frac{1}{2}\right) \tag{45}$$

which is equivalent to

$$0 < \alpha_1 + \alpha_2 < \alpha - n \max\left(\frac{1}{p_2}, 1 - \frac{1}{p_1}\right). \tag{46}$$

In region III we have $\lambda > 1/2$ and $\delta < \alpha$ and thus conclude $\alpha > n \max(1/p_2, 1 - 1/p_1)$ such that α_1 and α_2 in (46) may be suitably chosen. Consequently the theorem is proved assuming the upper estimates in the F-case to hold. It remains to verify this supposition.

Step 4. Dealing with the estimates from above in the F-case we rely on a partition of $\mathbb{R}^n$ into annuli up to a certain radius and a simultaneous control of the behaviour outside. For this purpose we make use of Corollary 3.1 several times. Now a_k always means $a_k(id^F)$. Let $l \in \mathbb{N}$ and $a_k^{(l)}$ be again the kth approximation number of the embedding $id^{(l)}: F^{s_1}_{p_1, q_1}(A_l) \to F^{s_2}_{p_2, q_2}(A_l)$, where $A_l = \{x \in \mathbb{R}^n : 2^{l-1} < |x| < 2^{l+1}\}$ for $l = 1, 2, \ldots$ and $A_0 = \{x \in \mathbb{R}^n : |x| < 2\}$ are defined as usual. We start considering region I. Then Corollary 3.1 and Proposition 2.3 give

$$a_k^{(l)} \leqslant c 2^{l\delta} k^{-\delta/n}. \tag{47}$$

In the sequel we always investigate suitable unions $\bigcup_{l=0}^{L} A_l$ in $\mathbb{R}^n$ and $L \in \mathbb{N}$ is chosen sufficiently large. We consider operators $B_l: f \mapsto f_{|A_l}$, $l = 0, 1, \ldots, L$, (in the sense of a suitable assigned resolution of unity) and get from the localization principle

$$\|B_l f \mid F^{s_1}_{p_1, q_1}\| \leqslant c 2^{-l\alpha} \|f \mid F^{s_1}_{p_1, q_1}(\alpha)\|. \tag{48}$$

We set

$$B^{L+1}: f \mapsto \left(id - \sum_{l=0}^{L} B_l\right) f \tag{49}$$

and have

$$\|B^{L+1}\| \leqslant c 2^{-\alpha L}. \tag{50}$$

Taking the additivity of approximation numbers into consideration (47)–(49) yield for $k=\sum_{l=0}^{L} k_l$

$$
\begin{aligned}
a_k^{\widehat{p_2}} = a_k^{\widehat{p_2}}\left(\sum_{l=0}^{L+1} B_l\right) &\leqslant c_1\left(\|B^{L+1}\|^{\widehat{p_2}} + \sum_{l=0}^{L} (a_{k_l}^{(l)})^{\widehat{p_2}}\, \|B_l\|^{\widehat{p_2}}\right)\\
&\leqslant c_2\left(2^{-L\alpha\widehat{p_2}} + \sum_{l=0}^{L} 2^{l\delta\widehat{p_2}} k_l^{-(\delta/n)\widehat{p_2}} 2^{-l\alpha\widehat{p_2}}\right)\\
&= c_2\left(2^{-L\alpha\widehat{p_2}} + \sum_{l=0}^{L} 2^{-l(\alpha-\delta)\widehat{p_2}} k_l^{-(\delta/n)\widehat{p_2}}\right)
\end{aligned}
\tag{51}
$$

where we used again the localization principle for F-spaces and denoted $\widehat{p_2}=\min(1,p_2)$. Let $\varepsilon>0$ and put $k_l = M2^{-l\varepsilon}$ for some $M>2^{L\varepsilon}$. (More precisely, we should choose constants c_l, $l=0,1,\ldots,L$, near 1 such that $k_l = c_l M 2^{-l\varepsilon}\in\mathbb{N}$, but we neglect this in the following as it causes no trouble.) Then (51) becomes

$$
\begin{aligned}
a_{c_1 M}^{\widehat{p_2}} &\leqslant c_2\left(2^{-L\alpha\widehat{p_2}} + M^{-(\delta/n)\widehat{p_2}} \sum_{l=0}^{L} 2^{-l(\alpha-\delta-\varepsilon(\delta/n))\widehat{p_2}}\right)\\
&\leqslant c_3 M^{-(\delta/n)\widehat{p_2}}
\end{aligned}
\tag{52}
$$

if L is chosen sufficiently large and $\varepsilon<n(\alpha-\delta)/\delta$. This procedure essentially uses $0<\delta<\alpha$. Thus (52) is the estimate in question

$$a_k \leqslant ck^{-\delta/n}.$$

The result for region VII now follows similarly as it did in the B-case, see (21). At this point we want to introduce a simplification. Regarding (51) and (52) the number $\widehat{p_2}$ has finally no influence at the result. Therefore we will always assume $\widehat{p_2}=1$ in the sequel though this is not quite true for $p_2<1$. But after all also this exponent cancels itself appearing on both sides.

Step 5. We care about region III now. Recall the already known homogeneity estimates, see [7: 5.4/4, 5] or [16: 2.2]

$$\|f(R\cdot) \mid F_{p_1,q_1}^{s_1}\| \leqslant cR^{s_1-n/p_1}\, \|f \mid F_{p_1,q_1}^{s_1}\|, \quad s_1 > n\left(\frac{1}{p_1}-1\right)_+, \quad R\geqslant 1 \tag{53}$$

and

$$\|f(R\cdot) \mid F_{p_2,q_2}^{s_2}\| \leqslant cR^{s_2-n/p_2}\, \|f \mid F_{p_2,q_2}^{s_2}\|, \quad s_2<0, \quad R\leqslant 1. \tag{54}$$

Applying these results to the annuli A_j we get for $s_1 > n(1/p_1 - 1)_+$ and $s_2 < 0$

$$a_k^{(j)}(F^{s_1}_{p_1,q_1}(A_j) \to F^{s_2}_{p_2,q_2}(A_j)) \leqslant c2^{j\delta}k^{-\lambda} \tag{55}$$

where we additionally used Proposition 2.3. Furthermore, we have $\langle x \rangle^\alpha \sim 2^{j\alpha}$ in A_j and hence

$$a_k^{(j)}(F^{s_1}_{p_1,q_1}(A_j, \langle x \rangle^\alpha) \to F^{s_2}_{p_2,q_2}(A_j)) \leqslant c2^{j(\delta-\alpha)}k^{-\lambda}. \tag{56}$$

The counterpart of (51) reads then as

$$a_k \leqslant c\left(2^{-\alpha L} + \sum_{j=0}^{L} 2^{j(\delta-\alpha)}k_j^{-\lambda}\right) \tag{57}$$

where we assumed $\widehat{p_2} = 1$. Then $k_j = M2^{-j\varepsilon}$, $\varepsilon > 0$, and a suitable choice of $\varepsilon < (\alpha - \delta)/\lambda$ results in

$$a_{c_1 M} \leqslant c_2(2^{-\alpha L} + M^{-\lambda}). \tag{58}$$

Assuming $L \geqslant \lambda/\alpha \log M$ we finally arrive at

$$a_k \leqslant ck^{-\lambda} \tag{59}$$

which is the desired result in region III under the additional assumptions $s_1 > n(1/p_1 - 1)_+$ and $s_2 < 0$. We will remove these restrictions by shifting the problem to an already known situation. The lift operator I_σ on S',

$$I_\sigma f = ((1 + |x|^2)^{\sigma/2} \hat{f})^\vee, \qquad \sigma \in \mathbb{R}, \tag{60}$$

maps $F^s_{p,q}$ isomorphically onto $F^{s-\sigma}_{p,q}$ (for details, see [13: 2.3.8]). This assertion extends to the spaces $F^s_{p,q}(\alpha)$, see [12: Chapter 5] and the references given there.

Suppose first $1 \leqslant p_1 < 2$, i.e. $n(1/p_1 - 1)_+ = 0$. We choose s_0 such that $s_2 < s_0 < s_1$ and $s_1' := s_1 - s_0 > n(1/p_1 - 1)_+$ and $s_2' := s_2 - s_0 < 0$. Then (59) together with $\lambda' = \lambda$ gives

$$a_k(F^{s_1'}_{p_1,q_1}(\alpha) \to F^{s_2'}_{p_2,q_2}) \leqslant ck^{-\lambda}$$

and hence (60) guarantees

$$a_k \leqslant ck^{-\lambda}.$$

The remaining case $0 < p_1 < 1$, i.e. $n(1/p_1 - 1) > 0$, is treated similarly.

Step 6. We handle the cases (iv) and (ix) of the main theorem now where $\lambda = (s_1 - s_2)/n - \max(1/2 - 1/p_2,\ 1/p_1 - 1/2) = \delta/n + \min(1/2 - 1/p_2,\ 1/p_1 - 1/2) > 1/2$, $0 < p_1 < 2 < p_2 < \infty$, $\delta > \alpha > 0$ and $s_2 < s_1$ are assumed to hold. We start dealing with case (iv). We apply the above proved result in region III for some $s_3 \in \mathbb{R}$,

$$s_2 < s_3 < s_1, \qquad \delta_1 = s_1 - \frac{n}{p_1} - \left(s_3 - \frac{n}{p_2}\right) < \alpha. \tag{61}$$

In particular, we split up our embedding in question

$$id(F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2}) = id(F^{s_3}_{p_2,q_2} \to F^{s_2}_{p_2,q_2}) \circ id(F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_3}_{p_2,q_2}) \tag{62}$$

where the embedding $F^{s_3}_{p_2,q_2} \to F^{s_2}_{p_2,q_2}$ is continuous. Then (59) applied to $F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_3}_{p_2,q_2}$ and s_3 chosen such that

$$\lambda_1 = \frac{\delta_1}{n} + \min\left(\frac{1}{2} - \frac{1}{p_2}, \frac{1}{p_1} - \frac{1}{2}\right) > \frac{1}{2} \tag{63}$$

together with $\delta_1 < \alpha$ finally yields for arbitrary $\varepsilon > 0$

$$a_k \leqslant c_\varepsilon k^{\ \alpha/n - \min(1/2 - 1/p_2,\ 1/p_1 - 1/2) + \varepsilon}, \tag{64}$$

i.e. the desired result in region IV. Here the assumption $\delta > \alpha > n \max(1 - 1/p_1, 1/p_2)$ becomes important for it guarantees the possibility to find $s_3 \in \mathbb{R}$ as described in (61) and (63), that is $\delta_1 < \alpha$ and $\lambda_1 > 1/2$.

Concerning region (ix) we follow the argumentation of the previous step and arrive at (57) now with $\delta > \alpha$. Choosing $k_j = M2^{j\varepsilon}$, $\varepsilon > 0$, yields (recall $\widehat{p_2} = 1$)

$$a_{c_1 M 2^{L\varepsilon}} \leqslant c_2 \left(2^{-\alpha L} + M^{-\lambda} \sum_{j=0}^{L} 2^{j(\delta - \alpha - \lambda\varepsilon)}\right) \tag{65}$$

which is for $\varepsilon > (\delta - \alpha)/\lambda > 0$

$$a_{c_1 M 2^{L\varepsilon}} \leqslant c_3 (2^{-\alpha L} + M^{-\lambda}). \tag{66}$$

Assuming $L \geqslant \lambda/\alpha \log M$ and afterwards the substitution $k = cM^{1 + (\lambda/\alpha)\varepsilon}$ leads to

$$a_k \leqslant c_\varepsilon k^{-\lambda\alpha/(\alpha + \lambda\varepsilon)}. \tag{67}$$

We remember $\varepsilon > (\delta - \alpha)/\lambda$ and hence

$$a_k \leqslant c_{\varepsilon'} k^{-\lambda\alpha/\delta + \varepsilon'} \tag{68}$$

for any $\varepsilon' > 0$. Looking again for the best possible λ and δ as above (in particular, we introduce again an additional parameter s_3 such that for δ_1 from (61) it holds $\delta_1 > n \max(1 - 1/p_1, 1/p_2) \geq \alpha$ and for λ_1 from (63) $\lambda_1 > 1/2$) we would have $\delta = n \max(1 - 1/p_1, 1/p_2)$ for $\lambda = 1/2$. Consequently (68) becomes then

$$a_k \leqslant c_\varepsilon k^{-\alpha/(2n \max(1 - 1/p_1, 1/p_2)) + \varepsilon} \tag{69}$$

for arbitrary $\varepsilon > 0$, i.e. the desired upper estimate in region IX.

Step 7. We concentrate on the regions II, V, VI and VIII now. The counterpart of (47) reads for $p_1 = p_2$ now

$$a_k^{(l)} \leqslant c 2^{l(s_1 - s_2)} k^{-(s_1 - s_2)/n}. \tag{70}$$

For $\delta > \alpha$ we determine k_l, $l = 0, ..., L$, by

$$k_l^{(s_1 - s_2)/n} = 2^{L(\alpha + \varepsilon)} 2^{l(s_1 - s_2 - \alpha - \varepsilon)} \tag{71}$$

where $\varepsilon > 0$ satisfies $\varepsilon < s_1 - s_2 - \alpha$. Hence

$$\sum_{l=0}^{L} k_l = 2^{Ln/(s_1 - s_2)(s_1 - s_2 - (s_1 - s_2 - \alpha - \varepsilon))} \sum_{l=0}^{L} 2^{l(s_1 - s_2 - \alpha - \varepsilon)n/(s_1 - s_2)}$$

$$= 2^{Ln} \sum_{l=0}^{L} 2^{(l - L)(s_1 - s_2 - \alpha - \varepsilon)n/(s_1 - s_2)} \leqslant c 2^{Ln} \tag{72}$$

and

$$\sum_{l=0}^{L} 2^{l(s_1 - s_2 - \alpha)} k_l^{-(s_1 - s_2)/n} = \sum_{l=0}^{L} 2^{l(s_1 - s_2 - \alpha - s_1 + s_2 + \alpha + \varepsilon)} 2^{-L(\alpha + \varepsilon)}$$

$$= 2^{-\alpha L} \sum_{l=0}^{L} 2^{\varepsilon(l - L)} \leqslant c 2^{-\alpha L} \tag{73}$$

and the counterpart of (51) (with $\widehat{p_2} = 1$) obviously results in

$$a_k \leqslant c k^{-\alpha/n}. \tag{74}$$

Now (74) leads almost directly to the upper estimates in (ii) and (viii). We choose s_0 as shown in Fig. 7 such that

$$s_0 - n/p_1 - (s_2 - n/p_2) = \delta - (s_1 - s_0) > 0$$

and

$$s_1 - s_0 > \alpha.$$

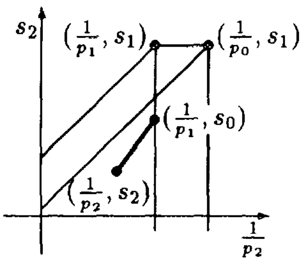

FIGURE 7

Then we have $F^{s_0}_{p_1,q_0} \subset F^{s_2}_{p_2,q_2}$ and (74) applied to $F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_0}_{p_1,q_0}$ yield together the upper estimates in region II and VIII.

We now deal with the regions V and VI. From (70) we have

$$a_k^{(l)}(F^{s_1}_{p_1,q_1}(A_l) \to F^{s_2}_{p_1,q_2}(A_l)) \leqslant c 2^{l(s_1-s_2)} k^{-(s_1-s_2)/n}.$$

Concerning the remaining embedding $F^{s_2}_{p_1,q_2}(A_l) \to F^{s_2}_{p_2,q_2}(A_l)$ for $p_2 < p_1$ we want to make use of Hölder's inequality. We proceed as in [5: 4.1.1] which is based on local means. Let $\psi_0 \in C_0^\infty(\mathbb{R}^n)$ be such that $\int \psi_0(x)\,dx \neq 0$, let $\psi = \Delta^N \psi_0$ for $N \in \mathbb{N}$ and introduce the local means

$$\psi(t,f)(x) = \int \psi(y)\, f(x+ty)\,dy, \qquad x \in \mathbb{R}^n, \quad t > 0 \tag{75}$$

and define $\psi_0(t,f)(x)$ similarly. Then we have for $2N > \max(s_2, n(1/p_1 - 1)_+)$ that for $f \in F^{s_2}_{p_1,q_2}(\mathbb{R}^n)$

$$\|\psi_0(1,f) \mid L_{p_1}(\mathbb{R}^n)\| + \left\| \left(\sum_{j=0}^{\infty} 2^{js_2q_2} |\psi(2^{-j},f)(\cdot)|^{q_2} \right)^{1/q_2} \Big| L_{p_1}(\mathbb{R}^n) \right\| \tag{76}$$

is an equivalent quasi-norm in $F^{s_2}_{p_1,q_2}(\mathbb{R}^n)$, for details see [14: 2.4.6]. By the usual extension-restriction procedure and Hölder's inequality for $p_2 < p_1$ we consequently get

$$\|id\colon F^{s_2}_{p_1,q_2}(A_l) \to F^{s_2}_{p_2,q_2}(A_l)\| \leqslant c 2^{nl(1/p_2 - 1/p_1)}. \tag{77}$$

Then (70) and (77) give

$$a_k^{(l)} \leqslant c 2^{l\delta} k^{-(s_1-s_2)/n} \tag{78}$$

both for the regions V and VI, $s_1 > s_2$ and $1/p_1 < 1/p_2 < 1/p_1 + \alpha/n$. Let first $\delta < \alpha$ and put $k_l = M2^{-l\varepsilon}$, $l = 0, ..., L$, for some $\varepsilon > 0$ and a constant $M > 2^{L\varepsilon}$. The counterpart of (51) (with $\widehat{p_2} = 1$) becomes

$$\begin{aligned} a_{c_1 M} &\leqslant c_2 \left(2^{-\alpha L} + M^{-(s_1 - s_2)/n} \sum_{l=0}^{L} 2^{l(\delta - \alpha + \varepsilon(s_1 - s_2)/n)}\right) \\ &\leqslant c_3 (2^{-\alpha L} + M^{-(s_1 - s_2)/n}) \leqslant c_4 M^{-(s_1 - s_2)/n} \end{aligned} \tag{79}$$

if $\varepsilon > 0$ is chosen sufficiently small, $\varepsilon < n(\alpha - \delta)/(s_1 - s_2)$, and $L \geqslant (s_1 - s_2)/(n\alpha) \log M$. Thus (79) gives the results in region V,

$$a_k \leqslant c k^{-(s_1 - s_2)/n}.$$

It now remains to prove the upper estimate in (vi). Let

$$\varkappa := \frac{(\delta - \alpha)\left(\dfrac{1}{p_2} - \dfrac{1}{p_1}\right)}{\left(\dfrac{\alpha}{n} - \dfrac{1}{p_2} + \dfrac{1}{p_1}\right)\left(\dfrac{\delta}{n} - \dfrac{1}{p_2} + \dfrac{1}{p_1}\right)}.$$

Then obviously $\varkappa > 0$ for $\delta > \alpha$ and $1/p_1 < 1/p_2 < 1/p_1 + \alpha/n$, $s_1 > s_2$. Furthermore $\delta - \alpha + \varkappa(s_1 - s_2)/n > 0$, for

$$\delta - \alpha + \varkappa \frac{s_1 - s_2}{n} = \delta - \alpha + \varkappa \left(\frac{\delta}{n} - \frac{1}{p_2} + \frac{1}{p_1}\right) = \frac{\alpha(\delta - \alpha)}{\alpha - \dfrac{n}{p_2} + \dfrac{n}{p_1}} > 0. \tag{80}$$

Let $k_l = M2^{-l\varkappa}$, $l = 0, ..., L$, then the counterpart of (51) reads as

$$\begin{aligned} a_{c_1 M} &\leqslant c_2 \left(2^{-\alpha L} + M^{-(s_1 - s_2)/n} \sum_{l=0}^{L} 2^{l(\delta - \alpha + \varkappa(s_1 - s_2)/n)}\right) \\ &= c_2 \left(2^{-\alpha L} + M^{-(s_1 - s_2)/n} 2^{L(\delta - \alpha + \varkappa(s_1 - s_2)/n)} \sum_{l=0}^{L} 2^{(l - L)(\delta - \alpha + \varkappa(s_1 - s_2)/n)}\right) \\ &\leqslant c_3 (2^{-\alpha L} + M^{-(s_1 - s_2)/n} 2^{L(\delta - \alpha + \varkappa(s_1 - s_2)/n)}) \end{aligned} \tag{81}$$

where we used the above mentioned properties of $\varkappa$. Substituting the above special $\varkappa$ we get for $L \geqslant 1/\alpha(\alpha/n - 1/p_2 + 1/p_1) \log M$

$$a_{c_1 M} \leqslant c_2 M^{-\alpha/n + 1/p_2 - 1/p_1}$$

what we just looked for in the region VI. This completes the proof.

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REFERENCES

1. C. BENNETT AND R. SHARPLEY, "Interpolation of Operators," Academic Press, Boston, 1988.
2. B. CARL AND I. STEPHANI, "Entropy, Compactness and the Approximation of Operators," Cambridge Univ. Press, Cambridge, UK, 1990.
3. D. E. EDMUNDS AND W. D. EVANS, "Spectral Theory and Differential Operators," Clarendon Press, Oxford, 1987.
4. D. E. EDMUNDS AND H. TRIEBEL, Entropy numbers and approximation numbers in function spaces, *Proc. London Math. Soc. (3)* **58** (1989), 137–152.
5. D. E. EDMUNDS AND H. TRIEBEL: Entropy numbers and approximation numbers in function spaces, II, *Proc. London Math. Soc. (3)* **64** (1992), 153–169.
6. J. FRANKE, Fourier-Multiplikatoren, Littlewood-Paley-Theoreme und Approximation durch ganze analytische Funktionen in gewichteten Funktionenräumen, Forschungsergebnisse Univ. Jena, N/86/8, 1986.
7. D. HAROSKE AND H. TRIEBEL, Entropy numbers in weighted function spaces and eigenvalue distributions of some degenerate pseudodifferential operators, I, *Math. Nachr.* **167** (1994), 131–156.
8. D. HAROSKE AND H. TRIEBEL, Entropy numbers in weighted function spaces and eigenvalue distributions of some degenerate pseudodifferential operators, II, *Math. Nachr.* **168** (1994), 109–137.
9. H. KÖNIG, "Eigenvalue Distribution of Compact Operators," Birkhäuser, Basel, 1986.
10. K. MYNBAEV AND M. OTEL'BAEV, "Weighting Functional Spaces and Differential Operator Spectrum," Nauka, Moscow, 1988. [in Russian]
11. A. PIETSCH, "Eigenvalues and s-Numbers," Akad. Verlagsgesellschaft Geest & Portig, Leipzig, 1987.
12. H.-J. SCHMEISSER AND H. TRIEBEL, "Topics in Fourier Analysis and Function Spaces," Wiley, Chichester, 1987.
13. H. TRIEBEL, "Theory of Function Spaces," Birkhäuser, Basel, 1983.
14. H. TRIEBEL, "Theory of Function Spaces, II," Birkhäuser, Basel, 1992.
15. H. TRIEBEL, "Interpolation Theory, Function Spaces, Differential Operators," North-Holland, Amsterdam, 1978.
16. H. TRIEBEL, A localization property for $B^s_{p,q}$ and $F^s_{p,q}$-spaces, *Studia Math.* **109**, No. 2 (1994), 183–195.

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